Homological algebra exercise sheet Week 4

1. Prove the following lemma stated during the lecture.

   Lemma 0.1 (Snake lemma). *Consider a commutative diagram of R-modules of the form*

$$\begin{tikzcd} & A' \arrow[r] \arrow[d, "f"] & B' \arrow[r, "p"] \arrow[d, "g"] & C' \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A \arrow[r, "i"] & B \arrow[r] & C. & \end{tikzcd}$$

   If the rows are exact, there is an exact sequence

$$\ker(f) \to \ker(g) \to \ker(h) \xrightarrow{\partial} \operatorname{coker}(f) \to \operatorname{coker}(g) \to \operatorname{coker}(h)$$

   with ∂ defined by the formula

$$\partial(c') = i^{-1}gp^{-1}(c'), \quad c' \in \ker(h).$$

   Moreover, if $A' \to B'$ is monic, then so is $\ker(f) \to \ker(g)$, and if $B \to C$ is onto, then so is $\operatorname{coker}(g) \to \operatorname{coker}(h)$.

2. Let $\mathcal{A}$ be an abelian category.

   (a) Let
$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$
   be a short exact sequence in $\mathbf{Ch}(\mathcal{A})$. Show that if two of the three complexes are exact, then so is the third.

   (b) Let $f_\bullet : C_\bullet \to D_\bullet$ be a morphism in $\mathbf{Ch}(\mathcal{A})$. Show that if $\ker(f_\bullet)$ and $\operatorname{coker}(f_\bullet)$ are acyclic, then $f_\bullet$ is a quasi-isomomorphism. Is the converse true?

3. Consider the following chain complex of abelian groups

$$\cdots \to \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \to \cdots .$$

   Show that it is acyclic, but not split exact.

4. Let $\mathcal{A}$ be an abelian category and consider a morphism $f_\bullet : C_\bullet \to D_\bullet$ in $\mathbf{Ch}(\mathcal{A})$.
   Show that $f_\bullet$ is null homotopic if and only if it extends to a map $(-s, f) : \operatorname{cone}(C_\bullet) \to D_\bullet$.